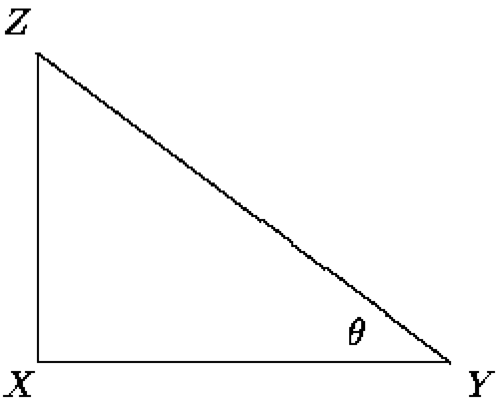
Heron's Formula

Heron or Hero was a Greek Mathematician who discovered a formula for the area of a triangle, $\triangle ABC$ with sides

$$a = BC, \;\; b = AC \;\; c = AB.$$

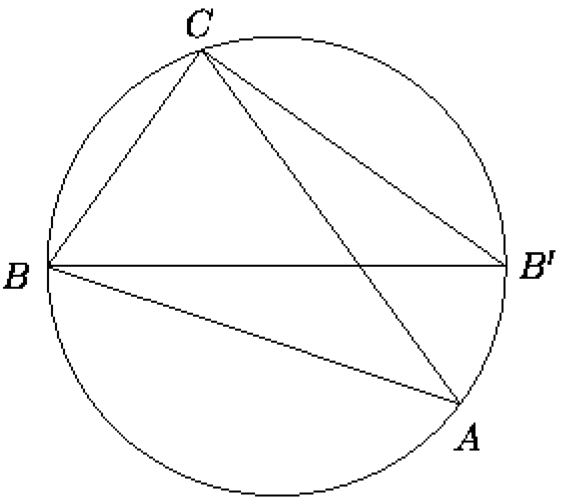
Before giving this formula, we need a little trigonometry. We assume that you are familiar with some trigonometry, and know what is meant by the ``sine'' and the ``cosine'' of an angle. Consider $\triangle XYZ$ with a right angle at X and let $\angle XYZ = \theta$.



We define $\sin(\theta) := \dfrac{XZ}{YZ}$ and $\cos(\theta) := \dfrac{XY}{YZ}$.

The Sine Rule

This give us the relation between the sides of a triangle, the sines of the opposite angles and the radius of the circumcircle. We consider $\triangle ABC$ and draw the circumcircle. Suppose that BB' is a diameter (of length $2R$).



Join CB', and note that

$$\angle A = \angle BAC = \angle BB'C.$$

Further, note that $\angle BCB'$ is a right angle. Observe that

$$\sin(BB'C) = \frac{BC}{BB'} = \frac{a}{2R},$$

where, as usual, we denote *BC* by *a*.Then

$$\frac{a}{\sin(A)} = \frac{a}{\sin(BB'C)} = 2R$$

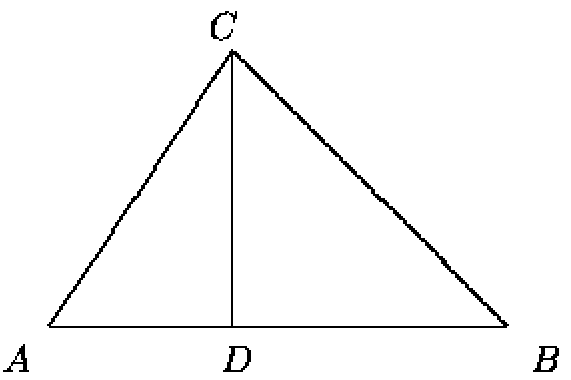
This shows that

$$\frac{a}{\sin(A)} = \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = 2R,$$

and so the Sine Rule is obtained.

The Cosine Rule

Consider $\triangle ABC$ and draw the perpendicular line to *AB* through *C*. Suppose this line meets *AB* at *D*.



Then

$$CD = b\sin(A) = a\sin(B)$$

so that

$$0 = b\sin(A) - a\sin(B),$$

and

$$AB = c = AD + DB = b\cos(A) + a\cos(B).$$

From this, by squaring the latter two equations, we get:

$$\begin{aligned} 0 &= b^2\sin^2(A) - 2ab\sin(A)\sin(B) + a^2\sin^2(B), \\ c^2 &= b^2\cos^2(A) + 2ab\cos(A)\cos(B) + a^2\cos^2(B). \end{aligned}$$

Adding these together gives

$$\begin{aligned} c^2 &= a^2 + b^2 + 2ab\left(\cos(A)\cos(B) - \sin(A)\sin(B)\right) \\ &= a^2 + b^2 + 2ab\cos(A+B) \\ &= a^2 + b^2 + 2ab\cos(C). \end{aligned}$$

This is the Cosine Rule.

A Further Observation

Since we have found that

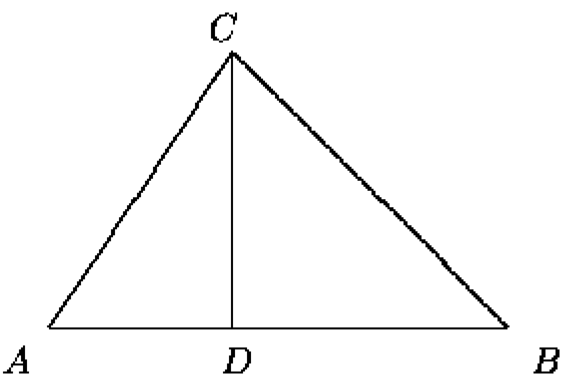
$$\cos(A) = \frac{b^2 + c^2 - a^2}{2bc},$$

we can deduce that

$$\begin{aligned} \sin^2(A) &= 1 - \cos^2(A) \\ &= 1 - \left(\frac{b^2 + c^2 - a^2}{2bc}\right)^2 \\ &= \frac{4b^2c^2 - (b^2 + c^2 - a^2)^2}{4b^2c^2} \\ &= \frac{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}{4b^2c^2} \qquad (1) \end{aligned}$$

The Area of a Triangle

We shall use the Sine Rule to get an new formula for the area of a triangle.



The area of $\triangle ABC$ is given by $\frac{1}{2}$ base $\times$ height. Thus

$$\begin{aligned} AREA\ \triangle ABC &= \frac{AB \times CD}{2} \\ &= \frac{c \times b\sin(A)}{2} \\ &= \frac{1}{2}bc\sin(A) \qquad\qquad (2) \\ &= \frac{1}{2}ac\sin(B) \\ &= \frac{1}{2}ab\sin(C) \\ &= \frac{abc}{4R} \end{aligned}$$

The perimeter of $\triangle ABC$ is given by *AB*+*BC*+*CA*. The semi-perimeter is half of this value! We denote the semi-perimeter by *s*, so that

$$s = \frac{a+b+c}{2}.$$

Heron's Formula

We use equations (1) and (2) to get

$$\begin{aligned} \sin^2(A) &= \frac{(a+b-c)(a-b+c)(-a+b+c)(a+b+c)}{4b^2c^2} \\ &= \frac{2(s-c)2(s-b)2(s-a)2s}{4b^2c^2} \\ &= \frac{4s(s-a)(s-b)(s-c)}{b^2c^2} \end{aligned}$$

A similar argument gives

$$\begin{aligned} \sin^2(B) &= \frac{4s(s-a)(s-b)(s-c)}{c^2a^2} \\ \sin^2(C) &= \frac{4s(s-a)(s-b)(s-c)}{a^2b^2}, \end{aligned}$$

and so

$$
\begin{aligned}
\sin(A) &= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc} \qquad\qquad (3)\\
\sin(B) &= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ca}\\
\sin(C) &= \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{ab},
\end{aligned}
$$

Using the formulae for the area of a triangle and these, we get

$$AREA\ \triangle ABC = \frac{1}{2}bc\sin(A) = \sqrt{s(s-a)(s-b)(s-c)}.$$

This is HERON's formula. The great advantage is that the area of the triangle can be calculated solely from the knowledge of the lengths of the sides. No information about the angles is required.

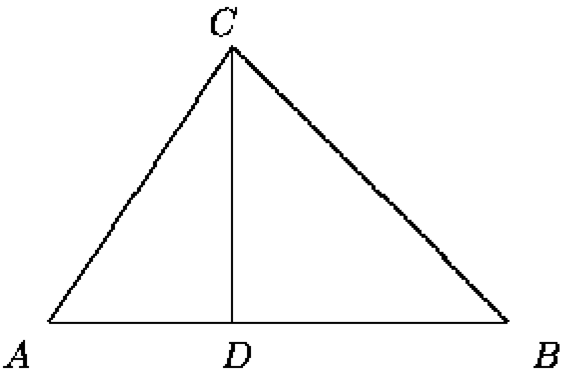
Useful Trigonometric Formulae

$$
\begin{aligned}
\sin(A \pm B) &= \sin(A)\cos(B) \pm \cos(A)\sin(B)\\
\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\
\cos(A \pm B) &= \cos(A)\cos(B) \mp \sin(A)\sin(B)\\
\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\
&= 2\cos^2(\theta) - 1\\
&= 1 - 2\sin^2(\theta)\\
&= \frac{1-\tan^2(\theta)}{1+\tan^{(}\theta)}\\
\tan(A \pm B) &= \frac{\tan(A) \pm \tan(B)}{1 \mp \tan(A)\tan(B)}\\
\tan(2\theta) &= \frac{2\tan(\theta)}{1-\tan^2(\theta)}
\end{aligned}
$$

$$
\begin{aligned}
\sin(\theta) \pm \sin(\phi) &= 2\sin\left(\frac{\theta \pm \phi}{2}\right)\cos\left(\frac{\theta \mp \phi}{2}\right)\\
\cos(\theta) + \cos(\phi) &= 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta + \phi}{2}\right)\\
\cos(\theta) - \cos(\phi) &= 2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{-\theta + \phi}{2}\right)\\
2\sin(\theta)\cos(\phi) &= \sin(\theta+\phi) + \sin(\theta-\phi)\\
2\cos(\theta)\sin(\phi) &= \sin(\theta+\phi) - \sin(\theta-\phi)\\
2\cos(\theta)\cos(\phi) &= \cos(\theta+\phi) + \cos(\theta-\phi)\\
2\sin(\theta)\sin(\phi) &= \cos(\theta-\phi) - \cos(\theta+\phi)
\end{aligned}
$$

Heron's Formula: Part II

Here we give an algebraic derivation of Heron's Formula.



Let *CD=h*, *AB=c*, *BC=a* and *CA=b* as usual. Then

$$c \;=\; \sqrt{a^2 - h^2} + \sqrt{b^2 - h^2},$$

so that

$$c^2 \;=\; a^2 + b^2 + 2h^2 + 2\sqrt{(a^2 - h^2)(b^2 - h^2)}$$

giving

$$2\sqrt{(a^2 - h^2)(b^2 - h^2)} \;=\; c^2 - a^2 - b^2 - 2h^2.$$

Squaring gives

$$4(a^2 - h^2)(b^2 - h^2) \quad \left(2h^2 + c^2 - a^2 - b^2\right)^2$$

or

$$4a^2b^2 - 4h^2\left(a^2 + b^2\right) + 4h^4 \;=\; 4h^4 + 4h^2\left(c^2 - a^2 - b^2\right) + \left(c^2 - a^2 - b^2\right)^2$$

or

$$\begin{aligned} 4h^2c^2 &= 4a^2b^2 - \left(c^2 - a^2 - b^2\right)^2 \\ &= \left(2ab - c^2 + a^2 + b^2\right)\left(2ab + c^2 - a^2 - b^2\right) \\ &= \left[(a+b)^2 - c^2\right]\left[c^2 - (a-b)^2\right] \\ &= (a+b+c)(a+b-c)(a-b+c)(-a+b+c) \\ &= (2s)(2s-2c)(2s-2b)(2s-2a) \end{aligned}$$

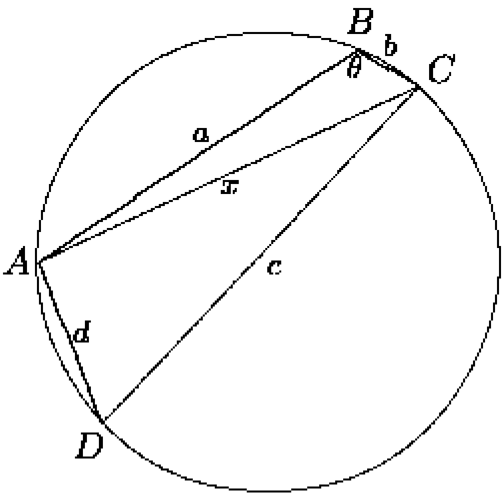
and Heron's formula follows at once.

Heron's Formula: Part III

Here we extend Heron's formula to find the area of *CYCLIC* quadrilaterals.Consider the cyclic quadrilateral *ABCD* and denote the sides and a diagonal as follows:

$$AB = a; \quad BC = b; \quad CD = c; \quad DA = d; \quad AC = x.$$

We define $s = \dfrac{a+b+c+d}{2}$. Let $\angle ABC = \theta$, so that $\angle CDA = 180^\circ$.



Then, the cosine rule, applied to $\triangle ABC$ and $\triangle CDA$, gives that

$$\begin{aligned} x^2 &= a^2 + b^2 - 2ab\cos(\theta) \\ &= c^2 + d^2 + 2cd\cos(\theta). \end{aligned}$$

Eliminating $\cos(\theta)$ from these equations gives

$$\frac{x^2 - a^2 - b^2}{2ab} + \frac{x^2 - c^2 - d^2}{2cd} = 0.$$

Thus

$$x^2\left(\frac{1}{2ab} + \frac{1}{2cd}\right) = \frac{a^2+b^2}{2ab} + \frac{c^2+d^2}{2cd},$$

yielding

$$x^2 = \frac{2cd\left(a^2+b^2\right) + 2ab\left(c^2+d^2\right)}{2ab + 2cd}.$$

Let Δ_1 represent the area of $\triangle ABC$ and Δ_2 represent the area of $\triangle CDA$.Heron's formula then gives us:

$$\begin{aligned} 16\,\triangle_1^2 &= (a+b+x)(a+b-x)(a-b+x)(-a+b+x) \\ &= \left[(a+b)^2 - x^2\right]\left[x^2-(a-b)^2\right]. \end{aligned}$$

Similarly we get that

$$16\,\triangle_2^2 = \left[(c+d)^2 - x^2\right]\left[x^2-(c-d)^2\right].$$

Now

$$\begin{aligned} [2ab+2cd]\left[x^2-(a+b)^2\right] &= 2cd(a^2+b^2)+2ab(c^2+d^2) \\ &\quad -2cd(a+b)^2 - 2ab(a+b)^2 \\ &= 2cd(-2ab)+2ab(c^2+d^2)-2ab(a+b)^2 \\ &= 2ab(c-d)^2 - 2ab(a+b)^2 \\ &= 2ab\left[(c-d)^2-(a+b)^2\right] \end{aligned}$$

Similarly

$$[2ab+2cd]\left[x^2-(a-b)^2\right] = 2ab\left[(c+d)^2-(a-b)^2\right]$$

Together, we now get

$$\begin{aligned} &[2ab+2cd]\left[x^2-(a+b)^2\right][2ab+2cd]\left[x^2-(a-b)^2\right] \\ &= 2ab\left[(c-d)^2-(a+b)^2\right]2ab\left[(c+d)^2-(a-b)^2\right] \\ &= 4a^2b^2(a+b+c-d)(a+b-c+d)(a-b+c+d)(-a+b+c+d) \end{aligned}$$

so that

$$\begin{aligned} 16\triangle_1^2[2ab+2cd]^2 &= 4a^2b^2\left[\left(\frac{s-a}{2}\right)\left(\frac{s-b}{2}\right)\left(\frac{s-c}{2}\right)\left(\frac{s-d}{2}\right)\right] \\ &\text{giving} \\ \triangle_1(2ab+2cd) &= \frac{ab}{8}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \\ &\text{Similarly} \\ \triangle_2(2ab+2cd) &= \frac{cd}{8}\sqrt{(s-a)(s-b)(s-c)(s-d)}. \end{aligned}$$

From this it is very easy to obtain the following formula for the area of a cyclic quadrilateral:

$$\square ABCD = \frac{1}{16}\sqrt{(s-a)(s-b)(s-c)(s-d)}.$$

Bruce Shawyer
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